Two Musical Orderings

Marcus Pendergrass*

May 24, 2013

The science of pure mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music.

Mathematics as an Element in the History of Thought
Alfred North Whitehead

Abstract

Orderings of one sort or another arise naturally whenever there is a notion of size or precedence among objects in a musical space. Paying attention to the order properties of the space can lead to new musical insights. In this paper we study two musical orders that illustrate this point. In the first, we show that many of the most familiar chord and scale types in Western music appear as extremal elements in certain partial orders induced by set inclusion on pitch class sets of T_n-type. In the second, we propose a family of partial orders for making timbral comparisons between musical tones. The ordering principle used is "unanimous agreement among informed listeners." We make this idea rigorous, and then study some of the basic properties of the partial orders that arise from it. Finally, we use these orders to compare the timbres of ten orchestral wind instruments in terms of their "brightness" and "flute-likeness." Our results show that these partial orders enable rigorous and fine-grained comparisons of timbres that are musically meaningful.

Keywords: partial order, quotient space, extremal element, pitch set class, *makam*, timbre, brightness.

Mathematics Subject Classifications: 06A99; 00A65

*Department of Mathematics and Computer Science, Hampden-Sydney College, Hampden-Sydney, Virginia, USA, email: mpendergrass@hsc.edu

1 Introduction

Orderings of various kinds are explicit or implicit in many of the concepts in music theory. Perhaps the most prominent examples of this are the recent mathematical theories of voice leading. In [1] Callender, Quinn, and Tymoczko lay down general principles that orderings should satisfy to qualify as reasonable measures of voice leading size (also see [2, 3]), while in [4] Hall and Tymoczko show that the general principles proposed in [1] are equivalent to the partial order of submajorization on the space of displacement multisets describing transitions from one chord to another. Indeed, orderings of one sort or another arise naturally whenever there is a notion of size or precedence among objects in a musical space. Paying careful attention to the order properties of the space can lead to new and interesting musical insights.

In this paper we study two partial orders that illustrate this point. The first, presented in section 2 below, is the partial order induced by set inclusion on the set classes of T_n-type from post-tonal theory. While this ordering is known to music theorists [5], we believe its explanatory power has not been fully appreciated. In particular, we will show that many of the most important scale and chord types in Western music appear as minimal elements in certain natural suborders of this partial order.

In section 3 we define a class of orderings that model comparisons of musical timbres. Timbre is a complex, multidimensional phenomenon, involving transient effects (e.g. attack, release), steady-state effects, as well as more complex psychoacoustic and even cultural effects [6, 7, 8, 9, 10, 11, 12]. One can question the extent to which even basic timbral qualities like "brightness" or "warmth" can be reduced to scalar quantities. Our approach here is to sidestep this issue by using partial orders to model judgements about various aspects of musical timbre. One judgement that musicians often make is that a certain instrument is "brighter" than another; for example, a trumpet is often thought to be brighter in tone than a horn. We provide a rigorous basis for such judgements using a partial order on an appropriately defined musical space. We will show that our approach generalizes to other aspects of timbre. Finally we apply these timbral partial orders to the timbres of ten orchestral wind instruments, and compare them with respect to their brightness and their "flute-likeness."

Proofs of all propositions in this paper are given in the final section.

2 Subset/Superset Order on Pitch Set Classes

One of the primary concerns of music theory is to understand the prevalence of specific musical forms and objects within a musical tradition. For instance, why is the diatonic scale so prominent in Western music? One mathematical approach to this question is to define a suitable space of all such musical objects, elucidate the structure this space, and then determine whether the musical objects in question occupy "privileged positions" in this space. In this section we show that many of the most prominent scale and chord types in Western music do in fact occupy privileged positions in certain partial orders on the space of pitch class sets.

The pitch classes under octave equivalence in the 12-tone system are identified with $\mathbb{Z}_{12}$, called pitch class space. Chords and scales correspond to subsets of $\mathbb{Z}_{12}$, called pitch class sets. Thus, the set of all pitch class sets is represented by $S = 2^{\mathbb{Z}_{12}}$, the power set of $\mathbb{Z}_{12}$. There are twelve distinct transpositions of pitch class space, namely $T_n x = x + n$ for $n, x \in \mathbb{Z}_{12}$, where the addition is modulo 12. These transpositions form a group $\mathcal{G}$ that is itself isomorphic to $\mathbb{Z}_{12}$. The action of $\mathcal{G}$ extends naturally to S; i.e. transpose a set by transposing each of its elements. The quotient space $S \,/\, \mathcal{G} = 2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$ identifies pitch class sets that are transpositionally equivalent. (So for instance all the diatonic collections are represented by a single equivalence class, all the octatonic collections are represented by another class, and so on.) The partial order of set inclusion on $2^{\mathbb{Z}_{12}}$ induces a relation $\preceq$ on $2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$ as follows: for $A, B \in 2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$, define $A \preceq B$ if and only if for all $a \in A$ there exists $b \in B$ such that $a \subseteq b$. The elements of $2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$ are precisely the set classes of T_n-type from post-tonal theory, and the induced relation is called the subset/superset ordering. [5, pp. 53, 96]. This construction obviously generalizes to N-tone equal temperament, and we have the following proposition:

Proposition 1. *The induced relation $\preceq$ is a partial order on $2^{\mathbb{Z}_N} \,/\, \mathbb{Z}_N$.*

Remark 1. The general situation here is that of a group $\mathcal{G}$ acting on a partial order $(S, \sqsubseteq)$ (not necessarily set inclusion), and the general question is whether the induced relation $\preceq$ is a partial order on $S/\mathcal{G}$. This question is relevant to other applications of partial orders in music theory (e.g. submajorization), and we address it briefly at the end of the paper.

A set class $A \in 2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$ will be represented as $A = \{0, a_1, a_2, \ldots, a_{n-1}\}$, where the pitch classes a_i are listed in increasing order. The interval from a_k to a_{k+1} will be called a scalar second, the interval from a_k to a_{k+2} is a scalar third, and so forth. (Index arithmetic is modulo n here.)

Denote the suborder of set classes whose scalar seconds span no more than k semitones by SS_k. It turns out that the minimal elements of the sub-orders SS_k are classes that have played prominent roles in compositional theory and practice, both in tonal and non-tonal music. First consider SS_2 as a suborder of $2^{\mathbb{Z}_{12}} \,/\, \mathbb{Z}_{12}$. The elements of SS_2 are set classes whose step sizes are either one or two semitones. These are precisely the classes that satisfy Tymoczko's "diatonic seconds" constraint in [13], and, as he points out, the minimal elements are the classes in SS_2 that contain no consecutive semitones. These are the whole tone collection, the diatonic collection, the octatonic collection, and the melodic minor (or acoustic) collection. Tymoczko goes on to observe that these four set classes can be characterized among all set classes in SS_2 by a condition on scalar thirds; namely, every scalar third in these scales spans either three or four semitones. The following proposition generalizes these ideas in the context of N-tone equal temperament.

Proposition 2. *Consider the partial order induced by set inclusion on $2^{\mathbb{Z}_N} \,/\, \mathbb{Z}_N$, and let SS_k be the suborder consisting of the set classes whose scalar seconds span no more than k semitones. Then a class $A \in SS_k$ is minimal in the suborder if and only if every scalar third in A spans at least $k+1$ semitones.*

Remark 2. We can view Proposition 2 as placing restrictions on the consecutive scale steps that a minimal set in SS_k can make. For instance, minimal elements of

Table 1: Minimal Elements of SS_k for 12-Tone Equal Temperament.

Suborder	Minimal Set Classes	Comment
SS_2	$\{0,1,3,4,6,7,9,10\}$	octatonic scale
	$\{0,2,3,5,7,9,11\}$	melodic minor scale
	$\{0,2,4,5,7,9,11\}$	diatonic scale
	$\{0,2,4,6,8,10\}$	whole tone scale
SS_3	$\{0,1,4,5,8,9\}$	a symmetric scale
	$\{0,2,4,6,8,10\}$	whole tone scale
	$\{0,2,4,7,9,11\}$	pentatonic scale
	$\{0,2,4,7,10\}$	dominant ninth chord
	$\{0,3,4,7,9\}$	a blues scale
	$\{0,3,4,7,10\}$	e.g. C7♯9
	$\{0,3,6,9\}$	diminished chord
SS_4	$\{0,3,6,9\}$	diminished chord
	$\{0,3,6,10\}$	half-diminished chord
	$\{0,3,7,10\}$	minor seventh chord
	$\{0,4,6,10\}$	e.g. C7♭5
	$\{0,4,7,10\}$	dominant seventh chord
	$\{0,4,7,11\}$	major seventh chord
	$\{0,4,8\}$	augmented triad
SS_5	$\{0,3,6,9\}$	diminished chord
	$\{0,3,7\}$	minor triad
	$\{0,4,6,10\}$	e.g. C7♭5
	$\{0,4,7\}$	major triad
	$\{0,4,8\}$	augmented triad
	$\{0,5,6,11\}$	a symmetric chord
	$\{0,5,10\}$	quartal triad

SS_3 must avoid consecutive steps of the forms $(1,1)$, $(1,2)$, and $(2,1)$; in addition to these, minimal elements of SS_4 must avoid $(1,3)$, $(2,2)$, and $(3,1)$, and so on.

Table 1 lists the minimal elements in each SS_k, $2 \leq k \leq 5$, for 12-tone equal temperament. Almost every set class in this table has played a prominent role in Western music. Of course, there are well-known scales that are not minimal in the sense described here: the harmonic minor, for example, or the Hungarian minor. Nevertheless, the overlap between the set classes that are minimal in our sense, and the scales and chords that are most common in Western music, is remarkable.

To what extent does this concept of minimality illuminate the scales and chords that appear prominently in non-Western musical traditions? The modal templates known as *makam* or *maqam* in traditional Turkish and Arabic art music would provide a rich source of data for a study addressing this question. Such a study would be complicated by the fact that the *makam* are actually used in practice as melodic types, and come along with intricate rules for composition and improvisation, some of which actually modify the pitches of certain notes in the *makam*. Nonetheless,

Table 2: *Makam* set classes in $N = 53$ equal temperament, along with associated SS classes and minimality.

Makam	Set Class	SS_k Class	Comment
hicaz	$\{0, 5, 17, 22, 31, 35, 39, 44\}$	SS_{12}	not minimal
rast	$\{0, 9, 17, 22, 31, 40, 48\}$	SS_9	minimal
segah	$\{1, 5, 14, 22, 31, 36, 45, 49\}$	SS_9	not minimal
kurdili hicazkar	$\{0, 4, 13, 22, 31, 35, 44\}$	SS_9	minimal
huzzam	$\{0, 5, 14, 19, 31, 36, 49\}$	SS_{13}	not minimal
nihavend	$\{0, 9, 13, 22, 31, 35, 44\}$	SS_9	minimal
hüseyni	$\{0, 8, 13, 22, 31, 39, 44\}$	SS_9	minimal
uşşak	$\{0, 8, 13, 22, 31, 35, 44\}$	SS_9	minimal
saba	$\{0, 8, 13, 18, 31, 35, 44, 49\}$	SS_{13}	not minimal

Bozkurt in [14] recommends representing the *makam* scale intervals of traditional Turkish art music using equal temperament with $N = 53$ pitch classes, and gives nominal scale intervals for nine of the most common *makamlar* (plural of *makam*) in that tradition. Table 2 lists these scales as pitch class sets in $2^{\mathbb{Z}_{53}}/\mathbb{Z}_{53}$, and indicates which are minimal in the appropriate suborders. As it turns out, five of these nine *makamlar* are indeed minimal. Note however that there are literally hundreds of *makamlar* used in the Turkish art song tradition, so Table 2 is quite incomplete. A more comprehensive study of the order relations among the modes from Turkish and Arabic music traditions would be very interesting indeed.

3 Timbral Partial Orders

Timbre refers to the *gestalt* of audible qualities, aside from pitch and intensity, that are associated with a musical sound. It is a complex phenomenon, with physical, psychological, and even cultural dimensions [6, 7, 8, 9, 10, 11, 12]. Musicians tend to speak of it in metaphorical terms: the trumpet is a "bright" instrument, as is the oboe, while the horn has a more "warm" or "mellow" sound. Such descriptions are necessarily inexact, and usefully so. Still, relatively unambiguous judgements about timbre are possible. For example, most "informed listeners" would agree that the trumpet has a brighter sound than the horn.

In the psychoacoustical literature, specific timbres are conceptualized as points lying in a "timbre space" of some sort, and the nature of this space is ascertained experimentally. (See [7], for example.) Typically, subjects are presented with pairs of sounds, and are asked to rate their dissimilarity on some scale. These dissimilarity ratings are interpreted as noisy distances between the timbres, and a multidimensional scaling analysis is used to determine the Euclidean space of lowest dimension that contains a set of points whose distances are approximately the same as the dissimilarity ratings. The dimensions of the timbre spaces in such studies turn out to be quite low, usually between two and four. Moreover, coordinates can be assigned to these spaces in a physically meaningful way, so that the location of a sound in timbre space can be predicted by mathematical descriptors of the associated waveform. One consistent result of these studies is that the spectral centroid of a sound correlates strongly with one of the dimensions of timbre space, which acousticians call "brightness" [10].

Here we take a different approach. Rather than modeling timbre space directly, we will model timbral qualities like brightness or warmth as partial orders on a set of waveform descriptors. There are two main motivations for our approach. In the first place, identifying a timbral quality like brightness with some numerical measure of spectral center, or indeed any scalar quantity, may be an oversimplification. Scalar quantities are linearly ordered. Is timbral brightness linearly ordered? For instance, must it necessarily be the case that either a bassoon is brighter than a bass flute, or *vice versa*? We might prefer a model that leaves open the possibility that certain timbres are simply incomparable in terms of brightness. Secondly, timbre is a complex, nuanced phenomenon, and is to some degree subjective. It is doubtful that any precise definition of a timbral quality like brightness would be accepted by all "informed listeners." We would like our model to acknowledge the fact that informed listeners may disagree in their judgements about a timbral quality like brightness.

We now propose such a model. In our model "timbral qualities" are defined by the set of their informed listeners. Informed listeners for a timbral quality need not agree precisely on the exact definition of the quality in question, but their judgements do conform to certain broad standards. The ordering criteria used for a given timbral quality is <u>unanimous agreement among informed listeners</u>. That is, given a timbral quality $\mathcal{V}$, and two sounds with descriptors $\mathbf{p}$ and $\mathbf{q}$, we will say that the sound represented by $\mathbf{q}$ has more of the timbral quality $\mathcal{V}$ than does $\mathbf{p}$ if and only if all informed listeners for $\mathcal{V}$ agree that this is the case. This partial order model for timbral qualities gives us a way to compare timbral qualities, without reducing them to scalars. It acknowledges the real-world fact that informed listeners for a given timbral quality may disagree in some of their judgements. In addition, the model allows for the formulation and study of a wide variety of timbral qualities.

We begin by describing a partial order of this type for timbral brightness. We then generalize this model to timbral qualities other than brightness. Finally we apply these models to the sounds of ten orchestral wind instruments. We compare these in terms of two timbral qualitities, their brightness, and their "flute-likeness."

3.1 An Ordering for Timbral Brightness

Mathematically, timbral brightness is associated with the presence of significant power in the higher harmonics of a sound. Therefore our comparisons will be made on the basis of the <u>discrete power spectrum</u> of a musical tone. Essentially, this amounts to the assumption that the sounds under consideration can be represented as Fourier series,

$$x(t) = \sum_n a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t), \tag{1}$$

where f_0 is the <u>fundamental frequency</u> (in Hertz) of the tone. Of course, this is a radical oversimplification. Real sounds have transient effects such as attack, vibrato, and release, as well as non-harmonic components (e.g. the "breathy" component of a flute sound), none of which are modeled in (1). Nevertheless, the Fourier series is a reasonable model of the harmonic steady-state aspects of timbre.

The discrete power spectrum of the signal x in (1) is the sequence

$$P_n = \frac{1}{2}\left(a_n^2 + b_n^2\right), \quad n \in \mathbb{N}.$$

Physically P_n represents the power of the signal x in the n^{th} harmonic of the fundamental f_0. For reasons that are beyond the scope of this paper, we will use the square-root of the power spectrum, known as the magnitude spectrum, as the waveform descriptor in making our comparisons. Since timbre is by definition independent of total volume or sound intensity, we normalize the magnitude spectrum so that it sums to 1. Hence, the timbre of the harmonic part x is represented by the sequence

$$p_n = C\sqrt{a_n^2 + b_n^2}, \quad n \in \mathbb{N},$$

where C is chosen so that $\sum p_n = 1$; i.e. $\mathbf{p} = (p_n)$ is a probability vector. We will refer to $\mathbf{p}$ as the timbral vector for the Fourier series (1). The set of all timbral vectors will be denoted by S.

We now define a partial order on the set of timbral vectors that models comparisons of timbral brightness. Rather than defining timbral brightness explicitly, let us simply say that an "informed listener" for timbral brightness is one who would agree that "higher harmonics are brighter than lower ones." Then we may model an informed listener for brightness as one who assigns a nonnegative "brightness value" v_n to harmonic n in such a way that $v_m \leq v_n$ whenever $m \leq n$. Such a listener is represented by her value vector $\mathbf{v} = (v_n)$, at least as far as judgements about brightness are concerned. (Listeners would have different value vectors for other timbral qualities.) Given a specific timbral vector $\mathbf{p}$, the listener assigns an overall brightness value to the timbre by computing a weighted average of the brightness of its harmonics, using the timbral vector for the weights. So the overall brightness value is

$$E_{\mathbf{p}}[\mathbf{v}] = \mathbf{p}^{\mathrm{T}}\mathbf{v} = \sum_n p_n v_n$$

This is simply the expected brightness value under the probability distribution that represents the timbre. An individual listener $\mathbf{v}$ judges the timbre $\mathbf{p}$ as no brighter than another timbre $\mathbf{q}$ if and only if $E_{\mathbf{p}}[\mathbf{v}] \leq E_{\mathbf{q}}[\mathbf{v}]$. Finally, if all informed listeners judge $\mathbf{p}$ as no brighter than $\mathbf{q}$, we will say that $\mathbf{p}$ precedes $\mathbf{q}$ in the "brightness order:"

$$\mathbf{p} \preceq \mathbf{q} \text{ if and only if } E_{\mathbf{p}}[\mathbf{v}] \leq E_{\mathbf{q}}[\mathbf{v}] \text{ for all } \mathbf{v} \in \mathcal{V}, \tag{2}$$

where $\mathcal{V} = \{\mathbf{v} : \mathbf{v} \geq 0, v_m \leq v_n \text{ if } m \leq n\}$ is the set of "informed brightness listeners."

Remark 3. It is straightforward to show that the brightness ordering is equivalent to

$$\mathbf{p} \preceq \mathbf{q} \text{ if and only if } \sum_{n \geq k} p_n \leq \sum_{n \geq k} q_n \text{ for all } k,$$

which is the familiar partial order of stochastic domination on probability vectors [16]. Note that $\sum_{n\geq k} p_n = \mathrm{Prob}_{\mathbf{p}}(N \geq k)$, where N is a random harmonic sampled from the probability distribution $\mathbf{p}$. Hence, we are saying that a timbre $\mathbf{q}$ is brighter than $\mathbf{p}$ if one is always more likely to sample a high harmonic from $\mathbf{q}$ than from $\mathbf{p}$.

Remark 4. Since the listener $\mathbf{v}$ defined by $v_n = n$ is an informed listener for brightness, it follows that $\mathbf{p} \preceq \mathbf{q}$ implies that $\sum_n np_n \leq \sum_n nq_n$. These sums are of course the means of the probability vectors $\mathbf{p}$ and $\mathbf{q}$, and represent measures of center for the magnitude spectra $\mathbf{p}$ and $\mathbf{q}$. Thus, the brightness ordering (2) is broadly consistent with the acoustical concept of brightness as spectral centroid. But (2) is a partial order: not all timbres are comparable in it. To reiterate: we consider this a strength of the model.

3.2 General Timbral Orders

We now generalize these ideas to timbral qualities other than brightness. To avoid issues of convergence, we will assume a maximum number M of harmonics. (Given the finite bandwidth of human hearing, this is not a limiting assumption.) The set of all harmonic, steady-state timbres is represented by the set S of probability vectors of length M. A listener is defined as a nonnegative vector $\mathbf{v}$ in $\mathbb{R}^M$; the n^{th} component of $\mathbf{v}$ represents the value the listener places on harmonic n in the timbre. A timbral quality is defined by its set of informed listeners. This set will be denoted by $\mathcal{V}$. We require that $\mathcal{V}$ be a cone; i.e. it is closed under nonnegative linear combinations. This models the idea that a listener can become informed about a timbral quality by adopting and mixing the values of other informed listeners. (Negative coefficients are not allowed because they would represent adopting values that are the opposite of those of an informed listener.) To avoid technicalities, we also require the cone to be topologically closed. An individual informed listener $\mathbf{v} \in \mathcal{V}$ judges timbre $\mathbf{p}$ to have less of the timbral quality $\mathcal{V}$ than timbre $\mathbf{q}$ if and only if $E_{\mathbf{p}}[\mathbf{v}] \leq E_{\mathbf{q}}[\mathbf{v}]$. Finally, we define the timbral order $\preceq$ on S for the timbral quality $\mathcal{V}$ by

$$\mathbf{p} \preceq \mathbf{q} \text{ if and only if } E_{\mathbf{p}}[\mathbf{v}] \leq E_{\mathbf{q}}[\mathbf{v}] \text{ for all } \mathbf{v} \in \mathcal{V}. \tag{3}$$

Hence, a timbre $\mathbf{p}$ precedes another timbre $\mathbf{q}$ in the order if and only if all informed listeners rate $\mathbf{p}$ as having less of the timbral quality than does $\mathbf{q}$.

Remark 5. Our model identifies a "timbral quality" with its nonnegative cone $\mathcal{V}$ of "informed listeners." Note that any $\mathbf{v} \in \mathcal{V}$ is a nonnegative multiple of some timbral (probability) vector in $\mathcal{V}_0 = \mathcal{V} \cap S$. Thus, timbral qualities may also be thought of as closed convex subsets $\mathcal{V}_0$ of timbral vectors in our model. The corresponding cones are of the form $\mathcal{V} = \{c\mathbf{v}_0 : c \geq 0,\ \mathbf{v}_0 \in \mathcal{V}_0\}$. The relation (3) is equivalent to $\mathbf{p} \preceq \mathbf{q}$ if and only if $E_{\mathbf{p}}[\mathbf{v}] \leq E_{\mathbf{q}}[\mathbf{v}]$ for all $\mathbf{v}_0 \in \mathcal{V}_0$.

The relation $\preceq$ defined by (3) is clearly reflexive and transitive; i.e. it is a preorder. The next proposition characterizes those cones $\mathcal{V}$ for which (3) defines a nontrivial partial order. (An order is said to be nontrivial if there exists a pair of distinct comparable elements.) The symbol $\mathbb{1}$ denotes the vector in $\mathbb{R}^M$ all of whose entries are 1. If A is a subset of $\mathbb{R}^M$, then $A^{\perp}$ denotes the subspace of $\mathbb{R}^M$ consisting of all vectors that are orthogonal to every vector in A.

Proposition 3. *Let $\mathcal{V}$ be a nonnegative closed cone in $\mathbb{R}^M$. Then* (3) *defines a partial order on the set S of probability vectors in $\mathbb{R}^M$ if and only if $\mathcal{V}^\perp \cap \mathbb{1}^\perp = \{\mathbf{0}\}$. The order is non-trivial if and only if $\mathbb{1}$ is not in the interior of $\mathcal{V}$.*

Example 1. The brightness order of section 3.1 is equivalent to (3), with $\mathcal{V}$ being the cone of nonnegative nondecreasing functions defined on $\{1, 2, \ldots, M\}$.

Remark 6. Our model declares $\mathbf{p} \preceq \mathbf{q}$ if and only if there is universal agreement among all informed listeners that this is the case. What if the agreement is not universal? In this case the proportion of informed listeners $\mathbf{v}$ who judge $\mathbf{p}$ to have less of the timbral quality $\mathcal{V}$ than $\mathbf{q}$ may be of interest. This proportion can be defined naturally as follows. Let V_0 be a random timbral vector, uniformly distributed on $\mathcal{V}_0 = \mathcal{V} \cap S$. Then the proportion of interest is $\text{Prob}\left(E_{\mathbf{p}}[V_0] \leq E_{\mathbf{q}}[V_0]\right)$. This probability can be approximated readily by simulation.

A case of special interest is when the cone $\mathcal{V}$ is the nonnegative span of a finite set of nonnegative vectors in $\mathbb{R}^M$. Let $\{\mathbf{v}_j : 1 \leq j \leq m\}$ be such a set, and let $\mathbf{V}$ be the M-by-m matrix whose j^{th} column is $\mathbf{v}_j$. Our cone is of the form $\mathcal{V} = \{\mathbf{Vc} : \mathbf{c} \geq \mathbf{0}\}$. If $\mathbf{p}$ and $\mathbf{q}$ are probability vectors, then the ordering (3) is equivalent to

$$\mathbf{p} \preceq \mathbf{q} \text{ if and only if } (\mathbf{q} - \mathbf{p})^{\mathrm{T}} \mathbf{V} \geq \mathbf{0}^{\mathrm{T}}. \tag{4}$$

The matrices $\mathbf{V}$ that yield nontrivial partial orders are characterized in the following proposition, which is a corollary to Proposition 3.

Proposition 4. *Let $\mathbf{V}$ be a nonnegative M-by-m matrix. Then* (4) *defines a partial order on the set S of probability vectors in $\mathbb{R}^M$ if and only if the rank of the augmented matrix $(\mathbf{V}\ \mathbb{1})$ is M. The order is nontrivial if and only if $\mathbf{Vc} \neq \mathbb{1}$ for all $\mathbf{c} > \mathbf{0}$.*

Example 2. The brightness order of section 3.1 is equivalent to (4), with the matrix $\mathbf{V}$ given by

$$\mathbf{V}(i,j) = \begin{cases} 1 & \text{if } M - i + 1 \leq j \leq M \\ 0 & \text{otherwise} \end{cases}$$

Two cones, or equivalently two convex subsets of S, are said to be <u>order-equivalent</u> if they generate the same partial order via (3). Given a cone $\mathcal{V}$, let $\mathcal{V}_0 = \mathcal{V} \cap S$, and consider the set $\mathcal{V}_0^*$ of all probability vectors lying on rays emanating from the uniform vector $\mathbf{u} = M^{-1}\mathbb{1}$ and containing some member of $\mathcal{V}_0$:

$$\mathcal{V}_0^* = \{(1-\lambda)\mathbf{u} + \lambda \mathbf{v}_0 : \lambda \geq 0,\ \mathbf{v}_0 \in \mathcal{V}_0\} \cap S$$

The cone determined by $\mathcal{V}_0^*$ is $\mathcal{V}^* = \{c\mathbf{v}_\mathbf{0}^* :\ c \geq 0,\ \mathbf{v}_\mathbf{0}^* \in \mathcal{V}_0^*\}$.

Proposition 5. *If $\mathcal{V}$ is a closed nonnegative cone in $\mathbb{R}^M$, then $\mathcal{V}$ and $\mathcal{V}^*$ are order-equivalent.*

Remark 7. Note that the uniform vector $\mathbf{u}$ is on the boundary of $\mathcal{V}_0^*$, and that $\mathcal{V}_0^*$ contains points on the boundary of S. This means that there are limitations on the kinds of timbral qualities that can be represented in this model. This is to be expected: some timbral qualities are not partially ordered.

3.3 Timbral Comparisons Among Ten Wind Instruments

We now apply these ideas to compare the timbres of ten wind instruments. We will consider three brass instruments (trumpet, trombone, horn), five reed instruments (oboe, bassoon, B♭ clarinet, E♭ clarinet, alto saxophone), and two flutes (bass flute, alto flute). The power spectra for these instruments were extracted from recordings made by Lawrence Fritts at the University of Iowa Electonic Studies [15]. All instruments were recorded playing the note E4 ($f_0 = 329.63$ Hertz) at a *mezzo forte* level. The sampling rate was 44100 Hertz. The power spectrum for each instrument was computed using the following procedure:

1. The steady-state portion of the waveform was identified manually, by inspecting the time-domain plot of the waveform.

2. To minimize envelope effects such as vibrato, the steady-state portion of the waveform was partitioned into snippets, each containing eight cycles of the fundamental frequency. The power spectrum for each of these snippets was then computed at frequency spacing $\Delta f = f_0$.

3. The overall harmonic, steady-state power spectrum was then computed by averaging the power spectra of the snippets.

Each power spectrum produced by this procedure has $\lfloor 22050/329.63 \rfloor = 66$ components. For all ten instruments, at least 99.99% of the total signal power is contained in the first twenty of these harmonics. In addition, for each instrument the harmonic with the least power among these twenty is between 66 to 80 decibels below the harmonic with the most power, meaning that all twenty harmonics are at potentially audible levels. Therefore, we choose to represent the harmonic, steady-state timbre of each instrument as a probability vector of length $M = 20$, obtained by normalizing these first twenty harmonics.

Figure 1 shows these ten instruments in the brightness order described in Section 3.1 above. The three maximal elements in the order are instruments that most musicians would be comfortable describing as bright: the alto saxophone, the trumpet, and the oboe. Naturally, as maximal elements these three instruments are incomparable with each other in terms of brightness. All three of the maximal elements are rated brighter than the horn, and all except the alto saxophone are brighter than the bassoon and the two clarinets. The bass flute and the bassoon are incomparable in terms of brightness, as are the two flutes, and the two clarinets. All these relationships seem musically reasonable, at least to this listener. (Keep in mind, however: informed listeners may disagree!)

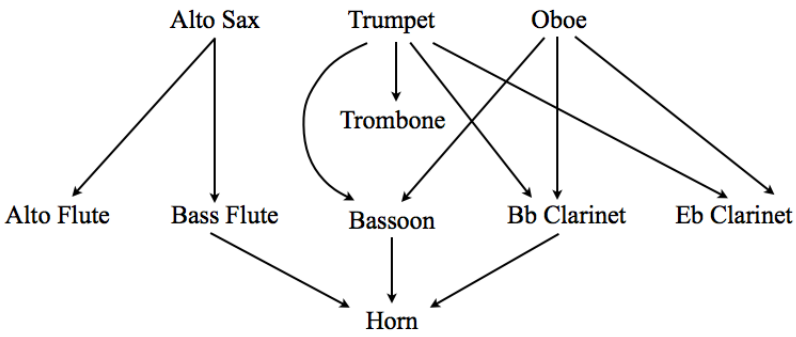

Figure 1. Ten wind instruments in the brightness ordering.

Nevertheless, the order does have some puzzling features from a musical point of view. For instance, the trumpet is not rated brighter than either of the flutes, the B♭ clarinet is rated as brighter than the horn but the E♭ clarinet is not, and the trombone is a minimal element in the order, and is incomparable to every instrument except the trumpet. Recall however that the ordering criteria underlying Figure 1 is quite strict: one instrument dominates another in the order only if there is unanimous agreement among all informed listeners for brightness that this is the case. As mentioned in Remark 6 of section 3.2, there is a natural definition for the proportion of informed listeners who rate one timbre as brighter than another, and this proportion can be approximated readily by simulation. Figure 2 below shows the brightness order of Figure 1 augmented by those pairs $(\mathbf{p}, \mathbf{q})$ of timbres for which at least 99% of informed brightness listeners agree that $\mathbf{q}$ is brighter than $\mathbf{p}$. (Each edge was computed using one million random informed brightness listeners.)

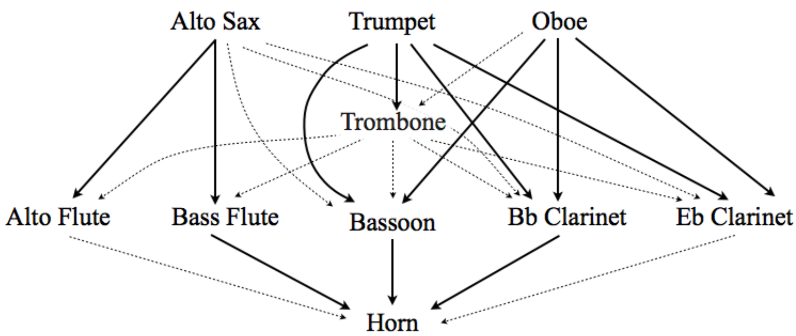

Figure 2. The augmented brightness order. Dashed arrows indicate that at least 99% of informed brightness listeners rate the initial node as brighter than the terminal node.

The augmented brightness order is more musically satisfying than the order of Figure 1, at least in this listener's opinion. Looking at the maximal elements, the trumpet and the oboe now dominate all the non-maximal elements, while the alto saxophone dominates all except the trombone. (Although in our simulation 95.2% of informed brightness listeners did rate the alto saxophone as brighter than the trombone.) The trombone dominates all the non-maximal elements, and the minimal element, the horn, is dominated by all the other elements in the order. It would be interesting to see whether the order of Figure 2 could be obtained by tweaking the definition of the cone that is used in the brightness ordering.

In addition to comparing these ten instruments in terms of brightness, we can use the ideas of section 3.2 to make comparisons between them in terms of other timbral qualities. Here, we compare the timbres on the basis of their "alto flute-likeness." Our "alto flute-like order" is defined as follows. Let $\mathbf{f} \in \mathbb{R}^M$ be the positive timbral vector for the alto flute instrument. Let $\rho \in \mathbb{R}$ be an arbitrary parameter, and let $\mathbf{v}_k = \mathbf{f} + \rho f_k \delta_k$, where f_k is the k^{th} component of $\mathbf{f}$, and δ_k is the k^{th} standard unit vector. The timbral vector $\mathbf{f}_k$ corresponding to $\mathbf{v}_k$ is a perturbation of the alto flute's timbral vector, principally in the k^{th} harmonic. One can easily show that the convex hull $\mathcal{V}_0$ of these timbral vectors contains the flute timbre $\mathbf{f}$ in its interior, and that the corresponding cone generates a nontrivial order via (3). Figure 3 below shows the rankings of our ten wind instruments in this order; $\rho = 1$ was used in this case.

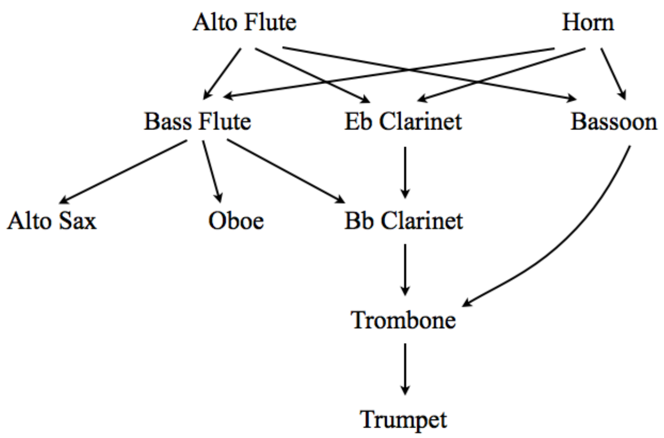

Figure 3. The ten wind instruments in the "alto flute-like order" ($\rho = 1$).

Again, the order makes a fair amount of sense musically. No instrument is more "alto flute-like" than the alto-flute itself (although this is not guaranteed theoretically). No instrument is closer to the alto flute in the order than the bass flute, and no instruments are less alto flute-like than the alto saxophone, the oboe, and the trumpet, with the trumpet being furthest removed from the alto flute in the order.

Letting ρ approach zero in the flute-like order results in a linear ordering of the ten wind instruments. (This is a generic property of orders of this type.) The limiting linear ordering is (in decreasing order): alto flute, horn, bass flute, E♭ clarinet, B♭ clarinet, alto saxophone, bassoon, trombone, oboe, and trumpet.

4 Proofs of the Main Results

We begin with a proof of Proposition 1. Though a direct proof is not difficult, we prefer to work in a somewhat more general setting, as it illuminates some of the key issues when working with partial orders on quotient spaces. Let $(S, \sqsubseteq)$ be a partially ordered set, and let $\mathcal{G}$ be a group of transformations mapping S into itself. An equivalence relation $\sim$ on S is defined by $a \sim b$ if and only if there exists a transform $T \in \mathcal{G}$ with $Ta = b$. We denote the equivalence class of $a \in S$ by $A = [a]$, and the set of distinct equivalence classes in S is denoted by $S \,/\, \mathcal{G}$. When does the order on S give rise to an order on $S \,/\, \mathcal{G}$?

Definition 1. The strong induced relation $\preceq_s$ on $S \,/\, \mathcal{G}$ is defined by

$$A \preceq_s B \text{ if and only if for all } a \in A \text{ there exists } b \in B \text{ such that } a \sqsubseteq b. \tag{5}$$

The weak induced relation $\preceq_w$ on $S \,/\, \mathcal{G}$ is defined by

$$A \preceq_w B \text{ if and only if there exists } a \in A \text{ and } b \in B \text{ such that } a \sqsubseteq b. \tag{6}$$

It is clear that if A precedes B in the strong relation, then it does so in the weak relation as well. Also, since the identity map is in $\mathcal{G}$, both the weak and strong relations are reflexive. But, in general, neither of these relations is an actual partial ordering of $S \,/\, \mathcal{G}$. Under certain conditions, however, they are.

Definition 2. The semigroup $\mathcal{G}$ is said to be increasing on the partial order $(S, \sqsubseteq)$ if for all T in $\mathcal{G}$, and all $a, b \in S$, whenever $a \sqsubseteq b$ in S, then $Ta \sqsubseteq Tb$ as well. The semigroup $\mathcal{G}$ is said to act transversely on S if for all $T \in \mathcal{G}$, and all $a \in S$, whenever $Ta \sqsubseteq a$, then in fact $Ta = a$.

Note that if $\mathcal{G}$ is a group that acts transversely on S, then a and Ta are always either incomparable, or identical. It follows in this case that all equivalence classes in $S \,/\, \mathcal{G}$ are antichains in the partial order on S.

Proposition 6. *Let $\mathcal{G}$ be a group acting on the partial order $(S, \sqsubseteq)$. Then:*

1. *The strong relation is a preorder on $S \,/\, \mathcal{G}$.*
2. *If $\mathcal{G}$ is increasing on S, then the strong and weak relations are identical.*
3. *If $\mathcal{G}$ acts transversely on S, then the strong relation is a partial order on $S \,/\, \mathcal{G}$.*

Proof.

1. It is obvious that the strong relation is transitive. We have already observed that it is reflexive, hence it is a preorder.
2. We have already observed that $A \preceq_s B$ implies $A \preceq_w B$. For the reverse implication, assume $A \preceq_w B$, and let a be an arbitrary element of A. Then there exist $a_0 \in A$ and $b_0 \in B$ with $a_0 \sqsubseteq b_0$. Since a and a_0 are in the same equivalence class, there exists $T \in \mathcal{G}$ with $Ta_0 = a$. Since T is increasing on S, we have $a \sqsubseteq Tb_0$. Since $Tb_0 \in B$, we conclude that $A \preceq_s B$.
3. To show the strong relation is a partial order, we need only verify that the antisymmetric property holds. Suppose that $A \preceq_s B$ and $B \preceq_s A$. Then there exists $a, a' \in A$ and $b \in B$ with $a \sqsubseteq b \sqsubseteq a'$. Since a and a' are in the same equivalence class, there exists $T \in \mathcal{G}$ with $Ta' = a$. Thus $Ta' \sqsubseteq a'$, and since $\mathcal{G}$ acts transversely, we have $a = a'$. Hence $b = a$, and thus $A = B$.

□

Note that if $\mathcal{G}$ is both increasing and acts transversely on S, then by Proposition 6 the weak and strong relations are identical, and form a partial order on $S\,/\,\mathcal{G}$. In such cases we will simply write "$\preceq$" for the induced partial order.

The following special case underlies Proposition 1. First, suppose the group $\mathcal{G}$ acts on a set S_0, and that S is some collection of subsets of S_0. Extend the action of $\mathcal{G}$ to S naturally, i.e.

$$Ta = \{Tx : x \in a\}, \quad a \in S,\ T \in \mathcal{G} \tag{7}$$

Let the partial order on S be given by set inclusion.

Corollary 1. *Let S be the collection of all finite subsets of some nonempty set S_0, and let the partial order on S be given by set inclusion. Let $\mathcal{G}$ be any group acting on S_0, and extend the action of $\mathcal{G}$ to S naturally via* (7)*. Then the induced relations* (5) *and* (6) *are identical, and form a partial order on $S\,/\,\mathcal{G}$.*

Proof. $a \subseteq b$ implies that $Ta \subseteq Tb$ for all $T \in \mathcal{G}$, so $\mathcal{G}$ is increasing on S. If $Ta \subseteq a$ but $Ta \neq a$ and a is finite, then by the pigeonhole principle there would have to exist distinct s_1 and s_2, both in a, with $Ts_1 = Ts_2$. But this would contradict the invertibility of the transform $T \in \mathcal{G}$. Hence, then inclusion cannot be strict, and thus $\mathcal{G}$ acts transversely on S. Now apply Proposition 6. □

Proof of Proposition 1. Apply Corollary 1 with $S_0 = \mathbb{Z}_N$, $S = 2^{\mathbb{Z}_N}$ and $\mathcal{G} = \mathbb{Z}_N$.

Proof of Proposition 2. If $A \in \mathrm{SS}_k$ has a scalar third spanning k or fewer semitones, then eliminate the middle pitch class in that third to produce a new set class B. This set class is still in SS_k, and $B \preceq A$ in the induced order. Hence A is not minimal. On the other hand, if every scalar third in A spans at least $k+1$ semitones, then eliminating the middle pitch class in any third results in a class B that is not in SS_k. Hence A is minimal in the suborder on SS_k.

Proof of Proposition 3. We need the following lemma, which is a basic fact from convex analysis. Its proof is included here for completeness.

Lemma 1. *Let E be a closed, convex subset of $\mathbb{R}^M$, and let $\mathbf{p} \in \mathbb{R}^M$. Then there exists a unique closest point $\mathbf{q}$ in E to $\mathbf{p}$. Moveover, the point $\mathbf{q}$ satisfies $(\mathbf{q}-\mathbf{p})^{\mathrm{T}}(\mathbf{v}-\mathbf{q}) \geq 0$ for all $\mathbf{v} \in E$.*

Proof. If $\mathbf{p} \in E$ we may take $\mathbf{q} = \mathbf{p}$ and we are done. Otherwise, since E is closed and convex, there exists a unique point $\mathbf{q} \in E$ that is closest to $\mathbf{p}$ (in the Euclidean metric). We need only show that $(\mathbf{q}-\mathbf{p})^{\mathrm{T}}(\mathbf{v}-\mathbf{q}) \geq 0$ for all $\mathbf{v} \in E$. Suppose by way of contradiction that this were not the case. Then there would exist $\mathbf{v} \in E$ such that $(\mathbf{q}-\mathbf{p})^{\mathrm{T}}(\mathbf{v}-\mathbf{q}) < 0$. Let $\mathbf{v}_\lambda = (1-\lambda)\mathbf{q} + \lambda\mathbf{v}$. By convexity $\mathbf{v}_\lambda \in E$ for all $\lambda \in [0,1]$ Let $f(\lambda) = \|\mathbf{p}-\mathbf{v}_\lambda\|^2$. A straightforward calculation shows that $f'(0) = 2(\mathbf{q}-\mathbf{p})^{\mathrm{T}}(\mathbf{v}-\mathbf{q}) < 0$, which contradicts the fact that $\mathbf{q}$ is the closest vector in E to $\mathbf{p}$. □

Proof of Proposition 3: Suppose $\mathcal{V}^{\perp} \cap \mathbb{1}^{\perp} = \{\mathbf{0}\}$. If $\mathbf{p}$ and $\mathbf{q}$ are probability vectors with $\mathbf{p} \preceq \mathbf{q}$ and $\mathbf{q} \preceq \mathbf{p}$, then $(\mathbf{p}-\mathbf{q})^{\mathrm{T}} \mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{V}$. It follows that $(\mathbf{p}-\mathbf{q}) \in \mathcal{V}^{\perp} \cap \mathbb{1}^{\perp}$, and thus $\mathbf{p} = \mathbf{q}$. Hence, the preorder $\preceq$ is a partial order. On the other hand, assume that $\preceq$ is a partial order, and let $\mathbf{w} \in \mathcal{V}^{\perp} \cap \mathbb{1}^{\perp}$. Since $\mathbf{w} \in \mathbb{1}^{\perp}$, there exist probability vectors $\mathbf{p}$, $\mathbf{q}$, and a constant c, such that $\mathbf{w} = c\,(\mathbf{p}-\mathbf{q})$. Since $\mathbf{w} \in \mathcal{V}^{\perp}$, $c\,(\mathbf{p}-\mathbf{q})^{\mathrm{T}} \mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{V}$. Thus, either $c = 0$, or $(\mathbf{p}-\mathbf{q})^{\mathrm{T}} \mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{V}$. In the first case $\mathbf{w}$ is clearly $\mathbf{0}$. In the second we have $\mathbf{p} \preceq \mathbf{q}$ and $\mathbf{q} \preceq \mathbf{p}$, implying $\mathbf{p} = \mathbf{q}$, and hence $\mathbf{w} = \mathbf{0}$.

We now show that the order is non-trivial if and only if $\mathbb{1}$ is not in the interior of $\mathcal{V}$. Suppose $\mathbb{1}$ is in the interior of $\mathcal{V}$. Let $\mathbf{p}$ and $\mathbf{q}$ be distinct probability vectors. Since $(\mathbf{q}-\mathbf{p})^{\mathrm{T}} \mathbb{1} = 0$, every open neighborhood of $\mathbb{1}$ contains points $\mathbf{v}$ with $(\mathbf{q}-\mathbf{p})^{\mathrm{T}} \mathbf{v} < 0$, and other points $\mathbf{v}$ with $(\mathbf{q}-\mathbf{p})^{\mathrm{T}} \mathbf{v} > 0$. Hence, $\mathbf{p}$ and $\mathbf{q}$ must be incomparable. Now suppose that $\mathbb{1}$ is not in the interior of $\mathcal{V}$. Then the uniform probability vector $\mathbf{u} = M^{-1}\mathbb{1}$ is not in the interior of $\mathcal{V}_0$. We first consider the case when $\mathbf{u} \notin \mathcal{V}_0$. By Lemma 1, there exists $\mathbf{q} \in \mathcal{V}_0$ that minimizes the distance to $\mathbf{u}$. Moreover, for this $\mathbf{q}$ we have $(\mathbf{q}-\mathbf{u})^{\mathrm{T}} (\mathbf{v}-\mathbf{q}) \geq 0$ for all $\mathbf{v} \in \mathcal{V}_0$. Therefore

$$(\mathbf{q}-\mathbf{u})^{\mathrm{T}} \mathbf{v} \geq (\mathbf{q}-\mathbf{u})^{\mathrm{T}} \mathbf{q} = (\mathbf{q}-\mathbf{u})^{\mathrm{T}} (\mathbf{q}-\mathbf{u}) \geq 0$$

for all $\mathbf{v} \in \mathcal{V}_0$. Therefore $\mathbf{u}$ and $\mathbf{q}$ are distinct comparable points in $\mathcal{V}$, and so the order is nontrivial. All that remains is to consider the case when $\mathbf{u}$ is on the boundary of $\mathcal{V}_0$. But this follows via an easy limiting argument from the case just proven.

Proof of Proposition 4. By Proposition 3 the preorder is a partial order if and only if $\mathcal{V}^{\perp} \cap \mathbb{1}^{\perp} = \{\mathbf{0}\}$. One readily sees that $\mathbf{w} \in \mathcal{V}^{\perp} \cap \mathbb{1}^{\perp}$ if and only if $\mathbf{w}^{\mathrm{T}} (\mathbf{V}\ \mathbb{1}) = \mathbf{0}^{\mathrm{T}}$. Hence in the present case $\mathcal{V}^{\perp} \cap \mathbb{1}^{\perp} = \{\mathbf{0}\}$ if and only the rank of $(\mathbf{V}\ \mathbb{1})$ is M. The interior of $\mathcal{V}$ consists of the vectors $\mathbf{Vc}$ as $\mathbf{c}$ ranges over all strictly positive vectors. Hence, by Proposition 3 the order is nontrivial if and only if $\mathbf{Vc} \neq \mathbb{1}$ for all $\mathbf{c} > \mathbf{0}$.

Proof of Proposition 5 . It is straightforward to show that $\mathcal{V}_0^*$ is convex, and that $(\mathbf{q}-\mathbf{p})^{\mathrm{T}} \mathbf{v}_0^* \geq 0$ for all $\mathbf{v}_0^* \in \mathcal{V}_0^*$ if and only if $(\mathbf{q}-\mathbf{p})^{\mathrm{T}} \mathbf{v}_0 \geq 0$ for all $\mathbf{v}_0 \in \mathcal{V}_0$. Therefore, $\mathcal{V}_0^*$ generates the same order as $\mathcal{V}_0$, and thus $\mathcal{V}^*$ and $\mathcal{V}$ are order-equivalent.

References

[1] C. Callender, I. Quinn, and D. Tymoczko, *Generalized voice leading spaces*, Science 320, 2008, pp. 346–348; available at http://dx.doi.org/10.1126/science.1153021.

[2] D. Tymoczko, *The geometry of musical chords*, Science 313, 2006, pp. 72 – 74; available at http://dx. doi.org/10.1126/science.1126287.

[3] D. Tymoczko, *Scale theory, serial theory, and voice leading*, Music Analysis, 27(1), 2008, pp. 1 – 49.

[4] R. Hall and D. Tymoczko, *Submajorization and the geometry of unordered sets*, Am. Math. Monthly 119(4), 2012, pp. 263 – 283; available at http://dx.doi.org/10.4169/amer.math.monthly.119.04.263

[5] J. Strauss, *Introduction to post-tonal theory*, 3rd ed., Pearson Prentice Hall, New York, 2005.

[6] J-C Risset and M. V. Mathews, *Analysis of musical instrument tones*, Physics Today, vol. 22, no. 2, pp. 23 – 30.

[7] Grey, John, *Multidimensional perceptual scaling of musical timbres*, J. Acoust. Soc. Am., 61(5), 1977, pp. 1270-1277

[8] V. Alluri and P. Tolviainen, *Effect of enculturation on the semantic and acoustic correlates of polyphonic timbre*, Music Percept., 29(3), 2012, pp. 297 – 310.

[9] C. Fales, *The paradox of timbre*, Ethnomusicology, 46(1), 2002, pp. 56 – 95.

[10] J. Marozeau, A. de Cheveigné, S. McAdams, and S. Winsberg, *The dependency of timbre on fundamental frequency*, J. Acoust. Soc. Am., 2003, 114(5), pp. 2946 – 2957.

[11] J. Marozeau and A. de Cheveigné, *The effect of fundamental frequency on the brightness dimension of timbre*, J. Acoust. Soc. Am., 121(1), 2007, pp. 383 – 387.

[12] W. Slawson, *The color of sound: a theoretical study in musical timbre*, Music Theory Spectrum, 3, 1981, pp. 132 – 141.

[13] D. Tymoczko, *Scale networks and Debussy*, J. Music Theory 48(2), 2004, pp. 219 – 294.

[14] A. C. Gedik and B. Bozkurt, *Evaluation of the makam scale theory of Arel for music information retrieval on traditional Turkish art music*, J. New Music Res., 38(2), 2009, pp. 103 – 116.

[15] L. Fritts, *Musical instrument samples*, http://theremin.music.uiowa.edu/MIS.html.

[16] M. Müller and M. Scarsini, *Stochastic order relations and lattices of probability measures*, SIAM J. Optimiz., 16(4), 2006, pp. 1024 – 1043.